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STATES OF EQUILIBRIUM AND SECONDARY LOSS OF STABILITY OF A STRAIGHT ROD LOADED BY AN AXIAL FORCE

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A general analytical solution of the problem of postcritical deformation of a straight incompressible rod loaded by an axial force is given. The bending of the rod is studied under various boundary conditions, and new states of equilibrium the occurrence of which is due to the secondary loss of stability are found. It is shown that, for simply supported and clamped rods, the solution bifurcates when the ends meet.

Introduction. Korobeinikov [1] showed that the secondary buckling of a simply supported rod occurs when the compressive load exceeds the Euler critical load. Kuznetsov and Levyakov [2, 3] studied the nonlinear deformation and stability of simply supported and clamped rods by a numerical method. They found bifurcation points and solution branches that were not known previously. It is, therefore, of interest to study analytically the postcritical behavior of the rod.

It is well known that the exact solution of the problem of strong plane bending of an incompressible elastic rod loaded by point forces and couples at its ends is written in terms of the elliptic integrals [4] or the Jacobi elliptic functions [5]. Love [5] considered the problem of determining the plane states of equilibrium of a rod loaded by end forces (elastica problem) and gave the general analytical solution of the problem under the assumption that the line of action of the resultant of forces is fixed. However, in using this solution to study the postcritical deformation of compressed straight rods, some difficulties arise, since in the cases of rod bending where a support reaction occurs, the direction of the resulting force acting on the rod end is unknown. Moreover, this solution fails to describe the states of equilibrium connected with the secondary loss of stability of rods, which were studied numerically in [2, 3].

In the present paper, the solution proposed in [5] is generalized to the case of nonsymmetric deformation of the rod, which enables one to obtain the results of [4, 5] and describe the solution branches found in [2, 3]. The problem of determining the singular points of the nonlinear solutions obtained is formulated.

1. General Solution of the Problem of Postcritical Deformation of a Compressed Rod. We consider the postcritical flexural deformation of an initially straight incompressible rod of length l and constant flexural rigidity EI compressed by the axial force P. The Cartesian coordinate system xOy is chosen in such a manner that the abscissa axis coincides with the axis of the undeformed straight rod and the coordinate origin is located at its left end. We denote the angle between the tangent to the elastic line and the Ox axis by β. The equation of equilibrium has the form

$$EI\,\frac{d^2\beta}{ds^2} + P\sin\beta - R\cos\beta = 0, \tag{1.1}$$

where s is the arc length of the rod ($0 \leqslant s \leqslant l$), R is the support reaction acting in the direction of the Oy axis, and the line of action of the force P coincides with the Ox axis. We assume that, in addition to the boundary conditions, the solution of Eq. (1.1) satisfies the relation

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$$\int_0^l \sin\beta\, ds = 0, \tag{1.2}$$

which is the condition of equal ordinates of the rod ends.

To integrate Eq. (1.1), we simplify it by expressing the end forces P and R in terms of the resultant force H:

$$P = H\cos\alpha, \qquad R = H\sin\alpha. \tag{1.3}$$

Here α is the angle between the direction of the resulting force H and the Ox axis. Generally, the angle α is an unknown quantity. Substituting (1.3) into Eq. (1.1), we find its first integral

$$\frac{EI}{2H}\Big(\frac{d\beta}{ds}\Big)^2 - \cos(\beta-\alpha) = C, \tag{1.4}$$

where C is an integration constant. We set

$$C = 2k^2 - 1, \tag{1.5}$$

where k is a parameter referred to as the modulus of elliptic integral. To construct a solution, we introduce the elliptic functions of modulus k and argument u [6]

$$u = \sqrt{\frac{H}{EI}}\, s + u_1, \qquad k\,\mathrm{sn}\, u = \sin\frac{\beta-\alpha}{2}, \qquad u_1 \leqslant u \leqslant u_2, \tag{1.6}$$

where the parameters u_1 and u_2 are determined from the boundary conditions at the rod ends and depend on the modulus k.

With the use of (1.5) and (1.6), we reduce Eq. (1.4) to the form

$$\beta' = 2k\,\mathrm{cn}\, u. \tag{1.7}$$

Hereafter, the prime denotes differentiation with respect to the variable u. The solution of Eq. (1.7) has the form

$$\beta = \theta + \alpha, \qquad \theta = 2\arcsin(k\,\mathrm{sn}\, u). \tag{1.8}$$

Integration of the relations dx/ds and dy/ds with allowance for (1.6) and (1.8) yields the parametric equation of elastica

$$\begin{gathered} x = \xi\cos\alpha - \eta\sin\alpha, \qquad y = \xi\sin\alpha + \eta\cos\alpha, \\ \xi = \sqrt{\frac{EI}{H}}\,[-u + u_1 + 2(E\,\mathrm{am}\, u - E\,\mathrm{am}\, u_1)], \qquad \eta = \sqrt{\frac{EI}{H}}\, 2k(\mathrm{cn}\, u_1 - \mathrm{cn}\, u), \end{gathered} \tag{1.9}$$

where $E\,\mathrm{am}\, u = \int_0^u \mathrm{dn}^2 u\, du$ is the incomplete elliptic integral of the second kind. The first two relations in (1.9) imply that the coordinate system of the rod xOy is rotated relative to the coordinate system of the elastica $\xi O\eta$ through the angle α (the $O\xi$ axis coincides with the axis of elastica compression [5]).

Integrating (1.2) with allowance for (1.6) and (1.8), we obtain the condition

$$2k(\mathrm{cn}\, u_1 - \mathrm{cn}\, u_2)\cos\alpha + [u_1 - u_2 - 2(E\,\mathrm{am}\, u_1 - E\,\mathrm{am}\, u_2)]\sin\alpha = 0. \tag{1.10}$$

Construction of a solution governing the bending of the rod for a given modulus k reduces to determination of three parameters u_1, u_2, and α from two boundary conditions at the rod ends and Eq. (1.10). The configuration of the rod is determined by expressions (1.9) where the resulting force is calculated by the formula $H = (u_2 - u_1)^2 EI/l^2$, which follows from the first relation (1.6) for $s = l$. Combining (1.3) and (1.6), we find the relation between the load P and the above-indicated parameters

$$P = (u_2 - u_1)^2 \cos\alpha\, \frac{EI}{l^2}. \tag{1.11}$$

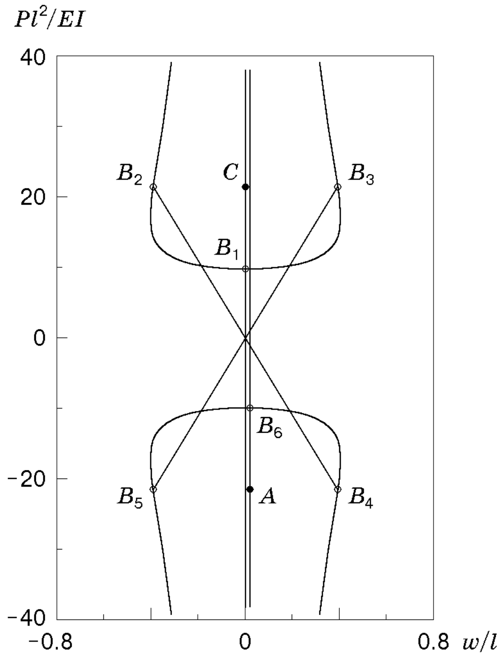

Fig. 1

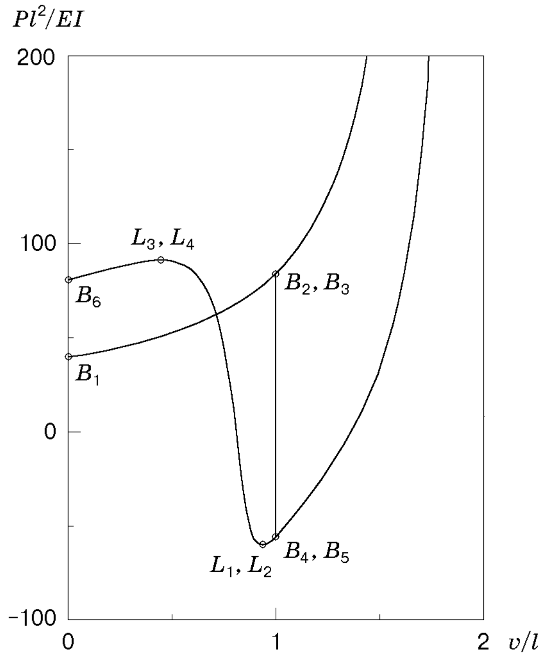

Fig. 2

2. Analysis of Solutions Under Different Boundary Conditions for Support of the Rod. Using relations given in Sec. 1, we consider the postcritical bending of the rod under different boundary conditions at its ends.

Case 2.1. *Simply Supported Ends.* By virtue of (1.7), the conditions for vanishing of the bending moments at $s = 0, l$ are written in the form $\operatorname{cn} u_1 = 0$ and $\operatorname{cn} u_2 = 0$, respectively. With allowance for the periodicity of the elliptic function, we obtain

$$u_1 = K, \qquad u_2 = (1 + 2n)K \qquad (n = \pm 1, \pm 2, \ldots), \tag{2.1}$$

where K is the complete elliptic integral of the first kind. By virtue of (2.1), condition (1.10) becomes

$$(K - 2E \operatorname{am} K) \sin \alpha = 0, \tag{2.2}$$

where $E \operatorname{am} K$ is the complete elliptic integral of the second kind. Condition (2.2) implies two types of solution, which are shown in Fig. 1 (see also Fig. 1 in [2]), where w is the mid-span deflection.

The first type of solution, which corresponds to the condition $\sin \alpha = 0$, is well known and describes the deformation of the rod after buckling in n semiwaves for $0 \leqslant k < 1$. In this case, the values of $\alpha = 0$ and $\alpha = \pi$ correspond to different solution branches, namely, the branches B_1C and $B_2B_1B_3$ refer to the value of $\alpha = 0$, and the branches B_6A and $B_4B_6B_5$ to the value of $\alpha = \pi$. To distinguish between the branches B_1C and B_6A, which refer to rectilinear configurations of the rod, we separate them in Fig. 1.

The first multiplier in (2.2) vanishes for $k_* = 0.9089085575 49$, $K_* = 2.321049732530$, and $E \operatorname{am} K_* = 1.160524866265$ and its vanishing is a condition under which the rod ends coincide [5]. In this case, the parameter α is arbitrary. The solution of the second type [see formula (1.8)] has the form $\beta = 2 \arcsin(k_* \operatorname{sn} u) + \alpha$ and describes the rotation of the deformed rod as a rigid body about the point of coincidence of its ends (closed branches $B_2B_4B_2$ and $B_3B_5B_3$).

Case 2.2. *One End is Simply Supported and the Other is Clamped.* The boundary conditions have the form

$$\beta'(u_1) = 0, \qquad \beta(u_2) = 0. \tag{2.3}$$

With allowance for (1.7), the first relation (2.3) is satisfied if

$$u_1 = K. \tag{2.4}$$

Substituting (1.8) into the second boundary condition (2.3), we express the angle α in the form $\alpha = -2\arcsin(k \operatorname{sn} u_2)$. Then,

$$\sin\alpha = -2k \operatorname{sn} u_2 \operatorname{dn} u_2, \qquad \cos\alpha = 1 - 2k^2 \operatorname{sn}^2 u_2. \tag{2.5}$$

Inserting (2.4) and (2.5) into (1.10), we obtain the following transcendental equation for u_2:

$$(1 - 2k^2 \operatorname{sn}^2 u_2) \operatorname{cn} u_2 + [K - u_2 - 2(E \operatorname{am} K - E \operatorname{am} u_2)] \operatorname{sn} u_2 \operatorname{dn} u_2 = 0. \tag{2.6}$$

We note that a similar equation was derived in [4] with the use of geometrical considerations. The solutions determined by the roots of Eq. (2.6) are known and describe the postcritical bending of the rod after its rectilinear equilibrium configuration loses stability. New solution branches were not found.

Case 2.3. *Both Ends are Clamped.* To analyze the states of equilibrium, we use Fig. 2, which shows the load P versus the change in distance between the rod ends v (see also Fig. 1 in [3]). We consider the boundary conditions

$$\beta(u_1) = 0, \qquad \beta(u_2) = 0. \tag{2.7}$$

Expressing $\sin\alpha$ and $\cos\alpha$ in terms of u_1 and u_2 with the use of relations (1.8) and (2.7) and substituting the resulting expressions into (1.10), we arrive at the following system of transcendental equations for u_1 and u_2:

$$\begin{gathered}(\operatorname{cn} u_2 - \operatorname{cn} u_1)(1 - 2k^2 \operatorname{sn}^2 u_1) + [u_1 - u_2 - 2(E \operatorname{am} u_1 - E \operatorname{am} u_2)] \operatorname{sn} u_1 \operatorname{dn} u_1 = 0, \\ (\operatorname{cn} u_2 - \operatorname{cn} u_1)(1 - 2k^2 \operatorname{sn}^2 u_2) + [u_1 - u_2 - 2(E \operatorname{am} u_1 - E \operatorname{am} u_2)] \operatorname{sn} u_2 \operatorname{dn} u_2 = 0.\end{gathered} \tag{2.8}$$

System (2.8) admits three types of solution.

The first type corresponds the case

$$\alpha = 0, \quad u_1 = 0, \quad u_2 = 4Kn \quad (n = \pm 1, \pm 2, \ldots), \tag{2.9}$$

where $0 \leqslant k < 1$. This solution describes the symmetric postcritical configurations of the rod (branch $B_2B_1B_3$ in Fig. 2).

The second type of solution of system (2.8) corresponds to the case where $\alpha \neq 0$ and $0 \leqslant k < 1$ and describes the nonsymmetric postcritical configurations of the rod with the inflection point in the middle section. In this case, the parameters u_1 and u_2 satisfy the relation $u_1 + u_2 = 2(2n-1)K$ $(n = \pm 1, \pm 2, \ldots)$. The branch $B_4L_1L_3B_6L_4L_2B_5$ (Fig. 2), which describes the development of the second buckling mode of a straight rod, corresponds to this type of solution.

The third type of solution of system (2.8) corresponds to the case where the rod ends coincide, i.e., when $k = k_*$. In this case, bearing in mind the periodicity of the elliptic function and using the expression (1.8) and boundary condition (2.7), we obtain

$$\alpha = -2\arcsin(k_* \operatorname{sn} u_1), \qquad u_2 = u_1 + 4K_* n \qquad (n = \pm 1, \pm 2, \ldots), \tag{2.10}$$

where u_1 is an arbitrary quantity. The first relation in (2.10) implies that the angle α varies from $-130.71°$ to $130.71°$. This type of solution is illustrated by the closed branch $B_2B_4B_5B_3B_2$ (see Fig. 2). Figure 3 shows configurations of the rod calculated for $u_1 = 0$, K_*, $2K_*$, and $3K_*$ (curves 1–4, respectively). Curve 5 is the trajectory of motion of the middle point of the rod upon deformation along the branch $B_2B_4B_5B_3B_2$ (see Fig. 2).

Case 2.4. *One End of the Rod is Free and the Other is Clamped.* The boundary conditions have the form $\beta'(u_1) = 0$ and $\beta(u_2) = 0$. In this case, it is necessary to set $\alpha = 0$ and ignore condition (1.10). With allowance for (1.7) and (1.8), the boundary conditions become

$$\operatorname{cn} u_1 = 0, \qquad \operatorname{sn} u_2 = 0. \tag{2.11}$$

Relations (2.11) are satisfied for $u_1 = K$ and $u_2 = 2nK$ $(n = \pm 1, \pm 2, \ldots)$. Solutions determined by these values are known. New solutions were not found.

3. Determination of the Singular Points of the Solution of the Problem of Postcritical Bending of a Rod. We formulate the problem of determining the singular points of the nonlinear solution describing the postcritical deformation of the rod. Replacing β and R in (1.1) by $\beta + \Delta\beta$ and $R + \Delta R$,

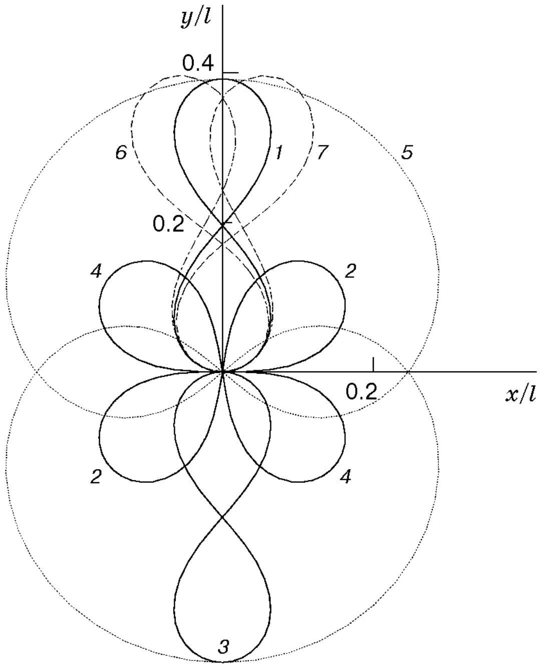

Fig. 3

respectively, and retaining only terms linear in the perturbed-state parameters $\Delta\beta$ and ΔR, with allowance for (1.3), (1.6), and (1.8), we obtain the equations

$$\Delta\beta'' + (1 - 2k^2\mathrm{sn}^2\, u)\Delta\beta = C_3[(1 - 2k^2\mathrm{sn}^2\, u)\cos\alpha - 2k\,\mathrm{sn}\, u\,\mathrm{dn}\, u\,\sin\alpha], \tag{3.1}$$

$$\cos\alpha\int\limits_{u_1}^{u_2}(1 - 2k^2\,\mathrm{sn}^2\, u)\Delta\beta\, du - 2k\sin\alpha\int\limits_{u_1}^{u_2}\mathrm{sn}\, u\,\mathrm{dn}\, u\Delta\beta\, du = 0, \tag{3.2}$$

where $C_3 = \Delta R/H$. We note that the differential equation similar to (3.1) was studied in [7] in a stability analysis of a circular ring compressed by radial forces.

The general solution of Eq. (3.1) has the form

$$\Delta\beta = C_1F_1(u) + C_2F_2(u) + C_3F_3(u), \tag{3.3}$$

where $F_1(u) = \mathrm{cn}\, u$, $F_2(u) = [E\,\mathrm{am}\, u - (1 - k^2)u]\,\mathrm{cn}\, u - \mathrm{sn}\, u\,\mathrm{dn}\, u$, and $F_3(u) = \cos\alpha + ku\,\mathrm{cn}\, u\sin\alpha$ (C_1 and C_2 are arbitrary constants). The derivatives of the functions $F_1(u)$, $F_2(u)$, and $F_3(u)$ are given by

$$\begin{gathered} F_1'(u) = -\mathrm{sn}\, u\,\mathrm{dn}\, u, \quad F_2'(u) = (k^2 - \mathrm{dn}^2\, u)\,\mathrm{cn}\, u - [E\,\mathrm{am}\, u - (1 - k^2)u]\,\mathrm{sn}\, u\,\mathrm{dn}\, u, \\ F_3'(u) = k(\mathrm{cn}\, u - u\,\mathrm{sn}\, u\,\mathrm{dn}\, u)\sin\alpha. \end{gathered} \tag{3.4}$$

Substituting (3.3) into (3.2) and calculating the quadratures, we obtain the condition

$$C_1J_1 + C_2J_2 + C_3J_3 = 0, \tag{3.5}$$

where

$$J_1 = \cos\alpha\,\mathrm{dn}\, u\,\mathrm{sn}\, u\Big|_{u_1}^{u_2} + k\sin\alpha\,\mathrm{cn}^2\, u\Big|_{u_1}^{u_2},$$

$$\begin{aligned} J_2 = {} & \cos\alpha\Big[E\,\mathrm{am}\, u\,\mathrm{dn}\, u\,\mathrm{sn}\, u + (\mathrm{dn}^2\, u - k^2)\,\mathrm{cn}\, u - (1 - k^2)u\,\mathrm{dn}\, u\,\mathrm{sn}\, u\Big]_{u_1}^{u_2} \\ & + k\sin\alpha\Big[E\,\mathrm{am}\, u\,\mathrm{cn}^2\, u - \mathrm{dn}\, u\,\mathrm{sn}\, u\,\mathrm{cn}\, u + (1 - k^2)u\,\mathrm{sn}^2\, u\Big]_{u_1}^{u_2}, \end{aligned} \tag{3.6}$$

$$J_3 = \cos^2\alpha(2E\,\mathrm{am}\,u - u)\Big|_{u_1}^{u_2} + k\sin\alpha\cos\alpha(3\,\mathrm{cn}\,u + \mathrm{dn}\,u\,\mathrm{sn}\,u)\Big|_{u_1}^{u_2} - \sin^2\alpha(E\,\mathrm{am}\,u - u\,\mathrm{dn}^2\,u)\Big|_{u_1}^{u_2}.$$

Substituting the general solution (3.3) into the boundary conditions at the rod ends and using condition (3.5), we obtain a system of three homogeneous equations for C_1, C_2, and C_3. Thus, the problem of determining the singular points of the solution reduces to finding the conditions under which the determinant of the system vanishes. Let us consider some conditions for support of the rod ends.

Case 3.1. *Simply Supported Rod.* Using the boundary conditions $\Delta\beta'(u_1) = 0$ and $\Delta\beta'(u_2) = 0$ and condition (3.5), with allowance for (2.1), (3.4), and (3.6), we obtain the system

$$\begin{gathered} C_1 + [E\,\mathrm{am}\,K - (1-k^2)K]C_2 = 0, \quad C_1 - (1+2n)[E\,\mathrm{am}\,K - (1-k^2)K]C_2 = 0, \\ (\mathrm{dn}\,K)C_1 + \mathrm{dn}\,K[E\,\mathrm{am}\,K - (1-k^2)K][(1+2n)(-1)^n - 1]C_2 + 2n(2E\,\mathrm{am}\,K - K)C_3 = 0, \end{gathered} \tag{3.7}$$

which yields the characteristic equation

$$[E\,\mathrm{am}\,K - (1-k^2)K](2E\,\mathrm{am}\,K - K) = 0. \tag{3.8}$$

One can show that the first factor in (3.8) is always positive and the second factor vanishes for $k = k_*$, i.e., when the rod ends coincide (see Case 2.1). According to (1.11) and (2.1), the secondary loss of stability occurs under the critical load $(P_n)_* = (2K_*n)^2EI/l^2$.

Solving system (3.7) for $k = k_*$, we infer that $C_1 = 0$, $C_2 = 0$, and C_3 is an arbitrary constant. According to (3.3), the function that describes the perturbed state of the rod has the form $\Delta\beta = C_3$, i.e., the buckling mode is rigid rotation of the deformed rod about the point where its ends meet. In Fig. 1, the points B_2, B_3, B_4, and B_5 are bifurcation points of the solution which correspond to the case $n = \pm 1$.

Case 3.2. *Clamped Rod.* We study the bifurcation of a solution of the first type (see Case 2.3). Using the boundary conditions $\Delta\beta(u_1) = 0$ and $\Delta\beta(u_2) = 0$ and relation (3.5), with allowance for (2.9), (3.3), and (3.6), we obtain the system

$$C_1 + C_3 = 0, \quad C_1 + 4n[E\,\mathrm{am}\,K - (1-k^2)K]C_2 + C_3 = 0, \quad (2E\,\mathrm{am}\,K - K)C_3 = 0. \tag{3.9}$$

The condition of existence of nontrivial solutions of system (3.9) implies the above-considered characteristic equation (3.8). Consequently, as in the simply supported case, a critical state of the bent rod with clamped ends occurs when its ends meet (see curves 1 and 3 in Fig. 3). The critical states are shown by the points B_2 and B_3 in Fig. 2 for $n = \pm 1$. It follows from (1.11) and (2.9) that the critical load for the secondary loss of stability is $(P_n)_* = (4nK_*)^2EI/l^2$.

Solving (3.9) for $k = k_*$, we find that $C_1 = -C_3$, $C_2 = 0$, and C_3 is an arbitrary constant. According to (3.3), the buckling mode of the bent rod is described by the function

$$\Delta\beta = C_3(1 - \mathrm{cn}\,u). \tag{3.10}$$

Linearizing Eqs. (1.6), (1.8), (2.9), and (3.10) and, then, using them to integrate the relations $d\tilde{x}/ds = \cos(\beta + \Delta\beta)$ and $d\tilde{y}/ds = \sin(\beta + \Delta\beta)$, we obtain the approximate formulas for the coordinates of the rod in perturbed states

$$\begin{gathered} \tilde{x} = \frac{l}{4nK_*}\Big[-u + 2E\,\mathrm{am}\,u + 2k_*C_3\Big(\frac{1}{2}\,\mathrm{sn}^2\,u + \mathrm{cn}\,u - 1\Big)\Big], \\ \tilde{y} = \frac{l}{4nK_*}\,[2k_*(1-\mathrm{cn}\,u) + C_3(-u + 2E\,\mathrm{am}\,u - \mathrm{dn}\,u\,\mathrm{sn}\,u)], \qquad 0 \leqslant u \leqslant 4nK_*. \end{gathered} \tag{3.11}$$

Figure 3 shows configurations of the rod (curves 6 and 7) calculated by formulas (3.11) for $n = 1$ and $C_3 = \pm 0.2$. These results show that the transition to a solution of the third type occurs at this bifurcation point (see Case 2.3).

Solutions of the other two types can be studied numerically.

The solution considered in Case 2.3 has four bifurcation points B_2, B_3, B_4, and B_5 (see Fig. 2) corresponding to the values of $u_1 = 0$, K_*, $2K_*$, and $3K_*$, respectively. The critical load $(P_n)_* = -(4nK_*)^2(2k_*^2 - 1)EI/l^2$ refers to the points B_4 and B_5.

An analysis of the solution describing the development of the second buckling mode of a straight rod (see Case 2.3) shows that, in addition to the well-known bifurcation point B_6, it contains the singular points L_1 (L_2) and L_3 (L_4), which are limit points [3], and the above-considered bifurcation point B_4 (B_5). Using the bisection method, we determined the critical loads $P = 92.038346EI/l^2$ ($k = 0.669523$) and $P = -60.838189EI/l^2$ ($k = 0.908084$) for the points L_1 (L_2) and L_3 (L_4), respectively.

A bending analysis of the rod one end of which is simply supported and the other end is clamped shows that the limit points exist for $k = 0.669523$ and $k = 0.908084$.

For a cantilevered rod, singular points were not found.

Conclusions. The problem of postcritical bending of a straight elastic rod loaded by an axial force has been considered. An analytical solution of the problem that generalizes the known elastica solution and describes nonsymmetric equilibrium configurations including those associated with the secondary loss of stability of rods has been given. The problem of determining the singular points of the solution obtained has been formulated and solved in the general form. To answer the questions concerning the type of singular points and the stability of the states of equilibrium, further investigations are required.

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